Memoryless Modulation

PAM

$s_m(t) = Re[A_m g(t) e^{j2\pi f_c t}] = A_m g(t) cos 2\pi f_c t \qquad m = 1, 2, 3, .., M$

$A_m = (2m - 1 - M)d \qquad Energy\ E_m = \frac{1}{2} A_m^2 E_g$

$P_{error} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \qquad P_b = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$

M-ary PAM

$error = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{d^2 E_g}{N_0}}\right)$

PSK

$s_m(t) = Re\left[g(t) e^{j2\pi \frac{(m-1)}{M}} e^{j2f_c t}\right] = g(t) cos\left[2\pi f_c t + \frac{2\pi}{M}(m-1)\right]$

$= g(t) cos \frac{2\pi}{M}(m-1) cos 2\pi f_c t - g(t) sin \frac{2\pi}{M}(m-1) sin 2\pi f_c t$

constant envelope $\qquad Energy\ E_m = \frac{1}{2} E_g$

Orthogonal Signaling FSK

$s_m(t) = Acos(2\pi [f_0 + m\triangle f] t) = Re\left[Ae^{2\pi f_0 t} e^{2\pi m \triangle f t}\right]$

$\triangle f_{bandpass\ min} = \frac{1}{2T} \qquad \triangle f_{lowpass\ min} = \frac{1}{T} \qquad distance = \sqrt{2E}\ or\ 2\sqrt{E}\ for\ biorthogonal$

Dimensionality theorem

N $\approx 2WT$ where W = bandwidth and T = duration

Union Bound for SNR > 4 ln2 $\qquad$ **Union Bound for SNR < 4 ln2**

$P_M < 2^k e^{-kE_b/2N_0} \qquad P_M < 2e^{-k(\sqrt{E_b}N_0 - \sqrt{ln2})^2}$

Given $\frac{E_b}{N_0} > 2ln2 = 1.39 = 1.42dB \qquad$ Given $\frac{E_b}{N_0} > ln2 = 0.693 = -1.6dB$

$P_e(M) \leq (M-1) Q(\sqrt{\frac{d_{min}^2}{2N_0}})$

Simplex Signaling

if $s(t) = \frac{1}{M}\sum_{i=1}^{M} s_i(t) \qquad s'_m(t) = s_m(t) - s(t) \qquad Energy\ E' = \frac{M-1}{M} E$

Memory Modulation

NRZI

$b_k = a_k\ xor\ b_{k-1}$ 10110001 $\rightarrow$ 0, 11011110

CPFSK

$s(t) = cos\left[2\pi f_0 t + 4\pi T f_d \int_{-\infty}^{t} v(\tau) d\tau\right] \qquad v(t) = \sum_n a_n g(t - nT_s)$

at any instant $f_i(t) = f_0 + f_d a_n \qquad h = 2Tf_d \qquad \theta_n = \pi h \sum_{k=-\infty}^{n-1} a_k$

$\theta(t; a) = 2\pi h \int_{-\infty}^{t} v(\tau) d\tau = \theta_n + 2\pi h a_n q(t - nT)$

MSK

$s(t) = cos(2\pi f_0 t + \frac{2\pi t}{4T} a_n + \theta - \frac{n\pi}{2} a_n)$ this is CPFSK with h = $\frac{1}{2}$

$\theta(t; a) = \frac{\pi}{2} \sum_{k=\infty}^{n-1} a_k + \pi a_n q(t - nT) \qquad \triangle f = \frac{1}{2T}$

Power spectra density

$S_x(f) = \frac{1}{T} S_I(f) |G(f)|^2 \qquad R_i(m) = \frac{1}{2} E[I_{n+m} I_n]$

Input Output signals

For a probabilistic signal let x = input and y = output

$S_y(f) = S_x(f)|H(f)|^2 \qquad S_{XY}(f) = S_X(f) H^*(f)$

$R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau)$

Match filter response

let s = the signal, the match filter would be $h(t) = s(T - t)$ and the response would be $y(t) = \int_0^t s(\tau)s(T - t + \tau)d\tau$

Noise due to match filter

$y(t) = \int_0^t r(\tau)h(t - \tau) = \int_0^t s(\tau)h(t - \tau) + \int_0^t n(\tau)h(t - \tau)d\tau = y_s(T) + y_n(T)$

variance of noise = $E[y_n^2(T)] = \frac{1}{2}N_2 \int_0^T h^2(T - t)dt$ $\quad SNR = \frac{y_s^2(T)}{E[y_n^2(T)]} = \frac{2E}{N_0}$

Frequency domain $H(f) = \int_0^T s(T - t)e^{-j2\pi ft}dt = S^*(f)e^{-j2\pi fT}$

MAP detection, MAX likelyhood detection

Let $r_1 = s + n_1$ and $r_2 = n_1 + n_2$ we want to find the best decision scheme such that $P(s = \sqrt{E}|r_1, r_2) > P(s = -\sqrt{E}|r_1, r_2)$ if we assume that both signals are equally likely we can use max likelyhood detection. $P(r_1, r_2|s = \sqrt{E}) > P(r_1, r_2|s = -\sqrt{E})$

$P(r_1 = s + n_1, r_2 = n_1 + n_2|s = \sqrt{E}) > P(r_1 = s + n_1, r_2 = n_1 + n_2|s = -\sqrt{E})$

$P(r_1 = \sqrt{E} + n_1, r_2 = n_1 + n_2|s = \sqrt{E}) > P(r_1 = -\sqrt{E} + n_1, r_2 = n_1 + n_2|s = -\sqrt{E})$

You want to rearrange the algebra to isolate the two noise signals since they are independent and can be split.

$P(n_1 = r_1 - \sqrt{E})P(n_2 = r_2 - n_1) > P(n_1 = r_1 + \sqrt{E})P(n_2 = r_2 - n_1)$

$P(n_1 = r_1 - \sqrt{E})P(n_2 = r_2 - r_1 + \sqrt{E}) > P(n_1 = r_1 + \sqrt{E})P(n_2 = r_2 - r_1 - \sqrt{E})$

From this you see that this becomes independent gaussian distributions which we can compare.

Information Theory

Individual information

$mutual\ information = I(x_i; y_j) = log\frac{P(x_i|y_i)}{P(x_i)}$

$self\ info = I(x_i) = -logP(x_i)$

$conditional\ info = I(x_i) = -logP(x_i|y_i)$

$I(x_i; y_j) = I(x_i) - I(x_i|y_j)$

Average information

$I(X;Y) = \sum_{i=1}^n \sum_{j=1}^m P(x_i, y_j)I(x_i, y_j)$ $\quad H(X) = -\sum_{i=1}^n P(x_i)I(x_i)$

$H(X|Y) = -\sum_{i=1}^n \sum_{j=1}^m P(x_i, y_j)P(x_i|y_j)$ $\quad I(X,Y) = H(X) - H(X|Y)$

$H(X_1, X_2, X_3...) = H(X_1) + H(X_2|X_1) + ...$

Rate distortion function

$R_g(D) = \frac{1}{2}log_2\frac{\sigma_x^2}{D}$ $\quad (0 < D < \sigma_x^2)$ $\quad R^*(D) \leq R(D) \leq= \frac{1}{2}log_2\frac{\sigma_x^2}{D}$

$R^*(D) = H(X) - \frac{1}{2}log_2 2\pi eD$ pg 109

Channel Capacity

$$C = max_{P(x_j)} \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} I(X,Y) = max_{P(x_j)} \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} p(x_j, y_i)log\frac{P(y_i|x_j)}{p(y_i)}$$

with power and bandwidth constrain, shannon's theorem

$C = Wlog\left(1 + \frac{P_{avg}}{WN_0}\right)$ since $P_{avg} = CE_b$ this could also be written as

$r = \frac{C}{W} = log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$ where r is measured in bits/second/Hz. If we solve for SNR we have

$\frac{E_b}{N_0} = \frac{2^{C/W}-1}{C/W} = \frac{2^r-1}{r}$ $\quad \frac{E_b}{N_0} = lim_{C/W->0}\frac{2^{C/W}-1}{C/W} = ln2 = -1.6dB$

This means that as SNR approach ln 2, the bandwidth required approaches infinity.

R_o Theorem , Error bounds

$P_{average\ error\ for\ random\ coding} \leq 2^{n(R_0-R)}$ $\quad$ $P(E|X_m) \leq \sum_{m'=1}^{M} \int p^{\lambda}(y|x_{m'})p^{1-\lambda}(y|x_m)dy$

Special case when $\lambda = \frac{1}{2}$ we have

$ED_{\lambda}(1->2) = \int(\sum_x p(x)\sqrt{P(y|x)})^2$

Special case when $\lambda = \frac{1}{2}$ $and\ p(x) = uniform$, $||x|| = Q$, $and\ p(x) = \frac{1}{Q}$

$ED_{\lambda}(1->2) = \frac{1}{Q^2}\int\left(\sum_{i=1}^{Q}\sqrt{p(y|x_i}\right)^2 dy$

$R_0 = 2log_2Q - log_2\int\left(\sum_{i=1}^{Q}\sqrt{p(y|x_i}\right)^2 dy$

R_0 for Binary Symmetric Channel = $log_2\frac{2}{1+2\sqrt{p(1-p)}}$

R_0 for AWGN BSC = $log_2\frac{2}{1+e^{-E_c/N_0}}$

Linear block code

You start off with a Generator matrix where $X_mG = Code$, where x are the bits, G is the transitional matrix. In the reduce form,$G = [I_k|P]$. We use the parity bit H to check for errors. $H = [-P|I_{n-k}]$

Hard code procedure

if $Y = C_{code} + e_{error}$ and $YH = (C_m + e)H' = eH' = S$ where S are the syndrom bits. There will be $2^{\#symdrombits}$ number of errors, we need to find all the error patterns that corresponds with the right syndrom. To find the error patterns, we first use up all the fewest bits, (1 bits) and move to different combinations of 2 and 3 .. bits. With this info, we can construct the standard array and the syndrom table. When we multiply H to Y, we get the syndroms and we look up the syndrom table for the possible error. We then subtract the error from the Code.

Rate relationship

coding gain = R_cd_{min}

d_{min} is the number of columns in the H matrix that are required to have an dependent columns.

if k = input bits and n = output bits $R_c = \frac{k}{n}$

E = E , $E_bk = E_cn$, $E_bR_c = E_c$

Probability error

where d^H_{min} is the Hamming distance

Soft decision $\quad P_e \leq (m-1)Q\left(\sqrt{2R_cd^H_{min}\frac{E_b}{N_0}}\right)$

Hard decision $\quad P_e \leq (M-1)[4p(1-p)]^{\frac{d^H_{min}}{2}}$

Hamming code(Type of systematic code)

$(n,k) = (2^m - 1, 2^m - 1 - m)$ where H (parity check) is the complete set of permutations

Convolutional Code

Transfer Function

-The transfer function is defined as $T(D) = \frac{X_e}{X_a}$ and when you solve for the equation you would end up with an expression.

-If you start from 000, a_d tells you the number of paths that has the distance d. So if you look below you have 1 branch with distance 6, 2 branches with distance 8 and etc.

$X_c = D^3X_a + DX_b \qquad X_b = DX_c + DX_d$
$X_d = D^2X_c + D^2X_d \qquad X_e = D^2X_b$
$T(D) = \frac{D^6}{1-2D^2} = D^6 + 2D^8 + 4D^10 + 8D^{12} + ... = \sum_{d=6}^{\infty} a_dD^d$

Probability error

if we have $T(D) = \sum_{d=dfree} a_dD^d$

$P_e \leq \sum_{d=d_{free}}^{\infty} a_d\frac{1}{2}e^{-R_cd\frac{E_b}{N_0}} = \frac{1}{2}T(D)|_{D=e^{-R_cd\frac{E_b}{N_0}}} \approx a_d\frac{1}{2}e^{-R_cd\frac{E_b}{N_0}}$

Trellis Coded Modulation

1. Draw constilation 2. label constilation with numbers sequencially 3. set partition n_1 times. 4.Use the size v convolutional code to decide # v states needed $\# = 2^{k(l-1)}$ where k is num v input and l is num of boxes in convolutional code. 5. Use set partition to randomly assign the trellis. (the branches from the trellis must have the same parents) 6. Convert the branch num into binary bits 7. notice that branches from the same state have equal bits, keep the equal bits and make the branches single branches. 8 Now we have the state transition trellis for the convolutional code.

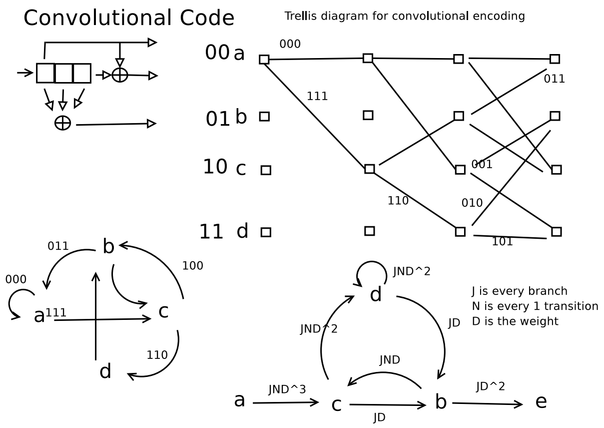

Figure 1: Convolutional code

Bandlimited Channels

$S = \sum_{n=0}^{\infty} I_ng(t-nT)$ -> C(t) + n(t) $= r(t) = I_nh(t-NT) + n(t)$
$h(t) = g(t) * c(t)$ and $y_k = I_k + \sum_{n=0}^{\infty} I_nX_{k-n} + V_k$
to avoid ISI we need to satisfy $\sum_{m=-\infty}^{\infty} X(f + \frac{m}{T}) = T$

SNR

$\frac{E_b}{N_0} = \frac{P_bT}{N_0} = \frac{P_b(1-\beta)}{N_0W}$

Raise Cosine function

$X_{rc}(f) = T \qquad 0 \leq f \leq \frac{1-\beta}{2T}$
$X_{rc}(f) = \frac{T}{2}\left\{1 + cos[\frac{\pi T}{\beta}(|f| - \frac{1-\beta}{2T})]\right\} \qquad \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T}$
$X_{rc}(f) = 0 \qquad |f| \geq \frac{1+\beta}{2T}$

Probability of error

$P_M = 1 - (1 - P_{\sqrt{M}})^2 \qquad P_{\sqrt{M}} = 2(1 - \frac{1}{\sqrt{M}})Q\left(\sqrt{\frac{3E_{avg}}{(M-1)N_0}}\right)$

The key equation to note is $\frac{1+\beta}{2T} = \frac{BW}{2}$ Where $\frac{1}{T}$ is equal to the symbol Rate

(R_S) measured in symbols/second. If you need to transmit 9600 bits/s with 8PSK,(3 bits) You would have $\frac{9600 bits/s}{3 bits/symbol}$. In general we want a β value that's greater than .5.

Example:

1. If we have 4000 Hz voice-bandpass channel, what's the bit rate if we use BPSK? (let beta = 1/2)

$\frac{1}{2T}(1 + 1/2) = 2000$ from this we get $\frac{1}{T} = 2666.666 symbol/sec$

Since this is BPSK with 1 bit per symbol this means we also have 2666 bits/sec

2. If we use 8QAM?

We know that the symbol rate is 2666 symbol/sec, with 3 bits we have 2666*3 = 8001 bits/sec

3. If we use 4FSK?

We divide the total bandwidth by 4 $\frac{4000Hz}{4} = \frac{1}{T} = 1000 symbol/sec$ with this value we can multiply by 2 bits to get the bit rate.